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# Adaptive Optimal Sampling Methodology for Reliability Prediction of Series Systems

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**Simulation-based system reliability prediction may require significant computations, particularly when the expected value of the system failure probability is relatively low. A methodology is presented for variance reduction of sampling-based series system reliability predictions based on optimal allocation of Monte Carlo samples to the individual failure modes. An algorithm is presented for adaptively allocating samples to member failure modes based on initial estimates of the member failure probabilities $p_i$. The methodology is demonstrated for a simple series system and a gas-turbine engine disk modeled using a zone-based series system approach. For the example considered, it is shown that the computational accuracy of the method does not appear to depend on the initial $p_i$ estimate. However, the computational efficiency is highly dependent on the initial $p_i$ estimate. The results can be applied to improve the efficiency of sampling-based series system reliability predictions.**

## Nomenclature

| | | |
|---|---|---|
| $\mathrm{COV}_i$ | = | coefficient of variation of failure probability for member $i$ |
| $E(\,)$ | = | mean value |
| $g$ | = | limit state function |
| $i, j$ | = | member number |
| $k_{\alpha/2}$ | = | standard normal variate |
| $m$ | = | number of members in system |
| $N$ | = | number of samples associated with risk prediction of system |
| $\boldsymbol{n}$ | = | vector containing $n_i$ for all $m$ members in system |
| $n_i$ | = | number of samples associated with risk prediction of member $i$ |
| $p_f$, $P_f$ | = | system failure probability, estimator of $p_f$ |
| $p_i$, $P_i$ | = | failure probability associated with member $i$, estimator of $p_i$ |
| $R_i$ | = | resistance of member $i$ |
| $S_i$ | = | applied stress for member $i$ |
| Var ( ) | = | variance |
| $Z(\boldsymbol{n})$ | = | objective function |
| $\gamma$ | = | relative sampling error |
| $\varepsilon$ | = | convergence tolerance |
| $\lambda$ | = | Lagrange multiplier |
| $\sigma_i$ | = | standard deviation of failure probability for member $i$ |
| $\phi\{\boldsymbol{n}, \lambda\}$ | = | Lagrange function |
| $1-\alpha$ | = | confidence level |

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## Introduction

AIRCRAFT gas-turbine engine rotors and disks may occasionally contain material anomalies that can form during the manufacturing process, which can lead to uncontained failure of the engine.[1] The occurrence of metallurgical anomalies is relatively rare, that is, an expected value on the order of one anomaly per million pounds of disk material,[2,3] and so a probabilistic approach has been developed for the fracture mechanics-based life prediction of components that may contain the anomalies.[4–8] A zone-based method is used to address the location uncertainty associated with these anomalies in which a component is subdivided into a number of zones of approximately equal risk. Component failure is modeled as a series system of zones, in which failure of any zone is interpreted as failure of the component.

Reliability prediction of general systems may require significant computations, particularly if correlation among members and postfailure material behavior are considered.[9,10] Although the time required for deterministic computations has been dramatically reduced by the availability of highly efficient computers, it is still a significant issue for simulation-based risk assessment of systems with relatively small failure probabilities. Over the past several decades, a number of approaches have been used to improve the efficiency of computations associated with system reliability predictions, such as bounding methods (e.g., Refs. 11 and 12) and various variance reduction techniques (e.g., Refs. 13–15, among many others). Many of these methods focus on reducing the number of simulations required to predict the overall system risk (at a specified level of accuracy), but provide little or no guidance regarding the number of simulations that should be allocated to predict the risk of individual members/zones. Over the past several years, a number of methods have been developed to improve the efficiency of zone-based system reliability predictions, such as tailored response surface,[16] zone refinement,[17] importance sampling,[16,18] and parallel processing,[19] among other improvements.

In this paper, a methodology is presented for variance reduction of sampling-based series system reliability predictions based on optimal allocation of Monte Carlo samples to the individual failure modes. It is an extension of a technique developed previously for mutually exclusive failure modes.[20,21] An algorithm is presented for adaptively allocating samples to member failure modes based on initial estimates of the individual member failure probabilities. The methodology is demonstrated for a simple series system and a gas-turbine engine disk modeled using the zone-based series system approach.

## Optimal Sampling Method for Series Systems

Consider the series system shown in Fig. 1 that consists of five members with an applied load. Illustrative values of the bivariant probability density functions (PDFs) for the individual member failure modes are shown in Fig. 2. Also shown is the member limit state

$$g(R_i, S_i) = R_i - S_i \tag{1}$$

In Fig. 2, note that members 1, 4, and 5 have a significant probability of occurrence in the failure region ($R_i \leq S_i$) and should have a larger contribution to system failure compared to members 2 and 3. An importance sampling strategy would place the samples near the limit state, that is, near members 1, 4, and 5. However, some samples should be provided for members 2 and 3 to quantify their contribution to system failure. The optimal allocation of samples is identified by minimizing the variance of the system failure probability, described in the equations that follow.

The failure probability $P_f$ of a series system of $m$ independent failure modes can be expressed as

$$P_f = 1 - \prod_{i=1}^{m} [1 - P_i] \tag{2}$$

where $(1 - P_i)$ is the survival probability associated with member $i$. The variance of $P_f$ is

$$\mathrm{Var}(P_f) = \prod_{i=1}^{m} E([1 - P_i]^2) - \left\{ \prod_{i=1}^{m} E[1 - P_i] \right\}^2 \tag{3}$$

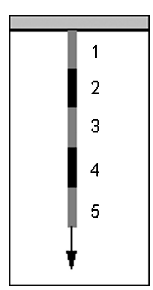


**Fig. 1 General five-member structural system under applied load.**

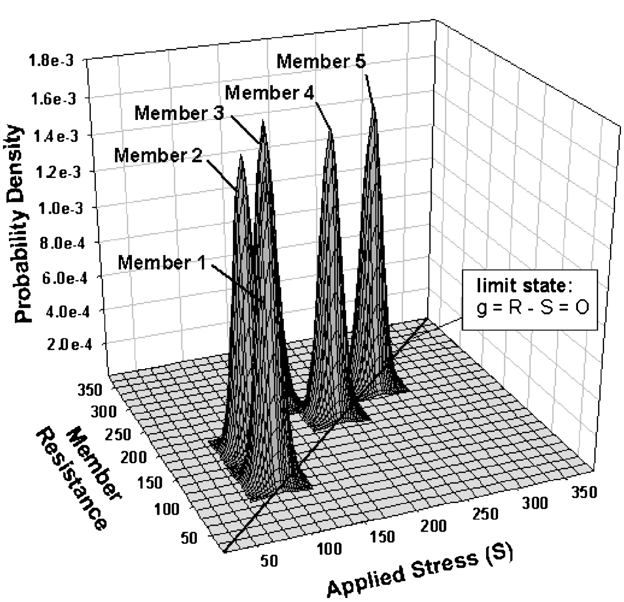


**Fig. 2 Bivariant PDFs revealing relative values of member failure probabilities in a system.**

The mean and variance of the member survival probability can be expressed as

$$E(1 - P_i) = 1 - p_i \tag{4}$$

$$\mathrm{Var}(1 - P_i) \simeq p_i(1 - p_i)/n_i \tag{5}$$

Given that

$$E\left([1 - P_i]^2\right) = (E[1 - P_i])^2 + \mathrm{Var}(1 - P_i) \tag{6}$$

Eq. (3) becomes

$$\mathrm{Var}(P_f) = \prod_{i=1}^{m} \left[ (1 - p_i)^2 + \frac{p_i(1 - p_i)}{n_i} \right] - \prod_{i=1}^{m} (1 - p_i)^2 \tag{7}$$

To identify values of $n_i$ that minimize $\mathrm{Var}(P_f)$, the following optimization formulation is used:

$$\min Z(n) = \prod_{i=1}^{m} \left[ (1 - p_i)^2 + \frac{p_i(1 - p_i)}{n_i} \right] - \prod_{i=1}^{m} (1 - p_i)^2 \tag{8}$$

subject to

$$N = \sum_{i=1}^{m} n_i \tag{9}$$

where $n_i$ is the number of Monte Carlo samples used to estimate the survival probability of member $i$, and $N$ is the total number of samples for the system.

Equations (8) and (9) provide a general expression for minimizing the variance of the system and can be solved using established optimization techniques. For example, Ref. 22 describes a nonlinear programming approach that can be used to minimize the variance of an experimental test design. The approach requires the use of a nonlinear programming solver that is applied to a simplified version of Eqs. (8) and (9).

Alternatively, Eqs. (8) and (9) can be solved directly using the Lagrange multiplier method,

$$\phi\{\boldsymbol{n}, \lambda\} = \prod_{i=1}^{m} \left[ (1 - p_i)^2 + \frac{p_i(1 - p_i)}{n_i} \right] - \prod_{i=1}^{m} (1 - p_i)^2 + \lambda \left( N - \sum_{i=1}^{m} n_i \right) \tag{10}$$

subject to the following conditions at the optimum:

$$\frac{\partial \phi}{\partial n_i} = 0, \qquad i = 1, 2, \ldots, m \tag{11}$$

$$\frac{\partial \phi}{\partial \lambda} = 0 \tag{12}$$

When these conditions are applied to Eq. (10), they become

$$\frac{\partial \phi}{\partial n_i} = \frac{-p_i(1 - p_i) \prod_{i=1}^{m} \left[ (1 - p_i)^2 + p_i(1 - p_i)/n_i \right]}{n_i^2 \left[ (1 - p_i)^2 + p_i(1 - p_i)/n_i \right]} - \lambda = 0 \tag{13}$$

$$\frac{\partial \phi}{\partial \lambda} = N - \sum_{i=1}^{m} n_i = 0 \tag{14}$$

When Eqs. (13) and (14) are combined,

$$n_i = \sqrt{\left\{p_i(1-p_i)\prod_{\substack{j=1\\ j\neq i}}^{m}\left[(1-p_j)^2 + p_j(1-p_j)/n_j\right]\right\}\Big/ -\lambda} \tag{15}$$

Substituting Eq. (15) into Eq. (14),

$$N = \sum_{i=1}^{m}\sqrt{\left\{p_i(1-p_i)\prod_{\substack{j=1\\ j\neq i}}^{m}\left[(1-p_j)^2 + p_j(1-p_j)/n_j\right]\right\}\Big/ -\lambda} \tag{16}$$

When Eqs. (15) and (16) are combined, the optimal number of samples for each member failure mode is given by

$$n_i = N\frac{\sqrt{p_i(1-p_i)\prod_{\substack{j=1\\ j\neq i}}^{m}\left[(1-p_j)^2 + p_j(1-p_j)/n_j\right]}}{\sum_{i=1}^{m}\sqrt{p_i(1-p_i)\prod_{\substack{j=1\\ j\neq i}}^{m}\left[(1-p_j)^2 + p_j(1-p_j)/n_j\right]}} \tag{17}$$

## Optimal Sampling Application: Five-Member Series System

To illustrate the methodology, Eq. (17) is used to identify the optimal number of samples for the five-member series system shown in Fig. 1. Member resistances $R_i$ are modeled as independent lognormal random variables, each subjected to a deterministic stress $S_i$. The main descriptors for these variables are indicated in Table 1, and associated probability densities are shown in Fig. 3. The limit state is given by Eq. (1) and can be solved analytically for each member failure mode. Analytical member failure probability results $P_i$ are indicated in Table 2.

Equation (1) can also be solved using numerical simulation with the variance of the member and system failure probabilities given by Eqs. (5) and (7), respectively. Suppose that 1 million Monte Carlo samples are used to predict the failure of the system. For uniform sampling, the same number of samples is allocated to the failure prediction of each member, that is, $n_i = 200{,}000$ for each member (Table 2). For optimal sampling, Eq. (17) is used to specify the number of samples for each member based on the analytical solution for $p_i$, indicated in Table 2. The standard deviation and coefficient of variation (COV) of the sampling-based member failure probability estimates, $\sigma_i$ and $\mathrm{COV}_i$, respectively, are given by

$$\sigma_i = \sqrt{p_i(1-p_i)/n_i} \tag{18}$$

$$\mathrm{COV}_i = \sigma_i/p_i \tag{19}$$

Values for these variables are indicated in Table 2. The variance of the system failure probability is also indicated in Table 2. When the values of the system failure probability $P_f$ are compared to the individual member failure probabilities $P_i$, note that members 1 and 2 contribute significantly to system failure. (More than 90% of system failure can be attributed to these two members.) When optimal sampling is used, the failure probability COV values of members 1 and 2 are smaller compared to the values for uniform sampling, leading to an overall reduction in the variance of the system failure probability. On the other hand, the failure probability COV is larger for the remaining members in the system when optimal sampling is used. (The member 5 failure probability COV value associated with optimal sampling is more then double the value associated with uniform sampling.) Probability densities for the $P_i$ estimates associated with the uniform and optimal sampling methods are shown in Fig. 4. The influence of optimal sampling on the PDF of $P_i$ is significant for all of the members considered, particularly for the strongest and weakest members, 5 and 1, respectively.

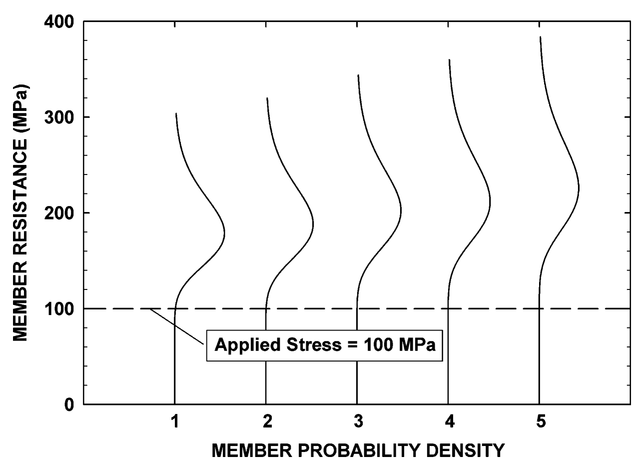


**Fig. 3 Probability densities associated with member resistances for five-member structural system shown in Fig. 1.**

**Table 1 Applied and ultimate stress values associated with five-member series system example**

| Variable | Unit | Name | Mean | Standard deviation | Distribution |
|---|---|---|---|---|---|
| $S$ | MPa | Applied stress | 100.0 | 0.0 | Deterministic |
| $R_1$ | MPa | Ultimate stress, member 1 | 190.0 | 38.0 | Lognormal |
| $R_2$ | MPa | Ultimate stress, member 2 | 200.0 | 40.0 | Lognormal |
| $R_3$ | MPa | Ultimate stress, member 3 | 215.0 | 43.0 | Lognormal |
| $R_4$ | MPa | Ultimate stress, member 4 | 225.0 | 45.0 | Lognormal |
| $R_5$ | MPa | Ultimate stress, member 5 | 240.0 | 48.0 | Lognormal |

**Table 2 Allocation of samples to members for uniform and optimal sampling strategies**

| | | Uniform sampling | | | Optimal sampling | | |
|---|---|---|---|---|---|---|---|
| Member | $P_i$ | $n_i$ | $\sigma_i$ | $\mathrm{COV}_i$ | $n_i$ | $\sigma_i$ | $\mathrm{COV}_i$ |
| 1 | $9.38E-04$ | 200000 | $6.84E-05$ | 0.07 | 442192 | $4.60E-05$ | 0.05 |
| 2 | $3.82E-04$ | 200000 | $4.37E-05$ | 0.11 | 282047 | $3.68E-05$ | 0.10 |
| 3 | $9.68E-05$ | 200000 | $2.20E-05$ | 0.23 | 141994 | $2.61E-05$ | 0.27 |
| 4 | $3.83E-05$ | 200000 | $1.38E-05$ | 0.36 | 89368 | $2.07E-05$ | 0.54 |
| 5 | $9.46E-06$ | 200000 | $6.88E-06$ | 0.73 | 44399 | $1.46E-05$ | 1.54 |
| System | $1.46E-03$ | 1000000 | $8.55E-05$ | 0.06 | 1000000 | $6.92E-05$ | 0.05 |

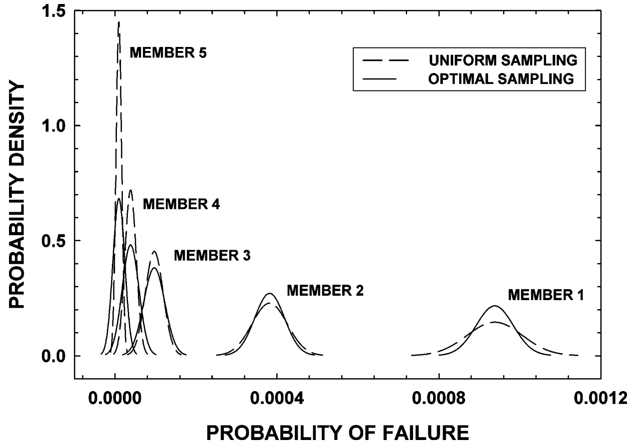


**Fig. 4 Probability densities associated with $p_i$ estimates for uniform and optimal sampling strategies.**

## Adaptive Optimal Sampling

Equation (17) indicates that the optimum number of samples for an individual member failure mode is dependent on knowledge of the failure probability $P_i$ for all of the member failure modes. However, because $p_i$ values are unknown before sampling, initial estimates of these values are required to make an estimate of the optimal number of samples in each zone. Initial estimates of the $p_i$ values can be obtained using a variety of numerical techniques, for example, numerical integration, moment-based methods, and numerical simulation. The efficiency and accuracy of the optimal sampling method is highly dependent on the number of samples $n_i$ used for the initial estimate of $p_i$. If $n_i$ is too large, the method is inefficient, that is, it provides no benefit over uniform sampling. On the other hand, if $n_i$ is too small, the method may be inaccurate. For example, if the number of samples for a member is so small that the limit state is not violated for any of the samples, then $p_i$ is estimated as equal to zero and $n_i$ is also zero. Accuracy of the initial estimate can be improved by setting a minimum value for $p_i$ when no samples violate the limit state, that is, $p_i = 1/n_i$. Depending on the limit state for a member, $p_i$ can sometimes be estimated using a nonsampling-based method, for example, numerical integration of dominant random variable(s), among others.

The total number of samples for the system $N$ can be determined by relating it to a desired sampling error and confidence interval. For a large number of samples $N$, the sampling-based estimate of the system failure probability $P_f$ is normally distributed by virtue of the central limit theorem. The associated confidence interval can be expressed as[23]

$$P\left(-k_{\alpha/2} < \frac{p_f - P_f}{\sqrt{p_f(1-p_f)/N}} \le k_{\alpha/2}\right) = 1 - \alpha \tag{20}$$

At the upper confidence limit, Eq. (20) can be expressed as

$$N = k_{\alpha/2}^2\left[p_f(1-p_f)\big/(p_f - P_f)^2\right] \tag{21}$$

Similar to Eq. (5), the variance of $P_f$ can be expressed as

$$\mathrm{Var}(P_f) \simeq p_f(1-p_f)/N \tag{22}$$

If the individual $p_i$ values are relatively small, the variance of $P_f$ can be approximated as

$$\mathrm{Var}(P_f) \simeq \sum_{i=1}^{m} \mathrm{Var}(P_i) \simeq \sum_{i=1}^{m} \frac{p_i(1-p_i)}{n_i} \tag{23}$$

The sampling error can be defined as

$$\gamma = (p_f - P_f)/P_f \tag{24}$$

When Eqs. (21–24) are combined, the number of samples for a specified sampling error and confidence interval can be estimated as[20,21]

$$N = \frac{k_{\alpha/2}^2}{\gamma^2 P_f^2}\left[\sum_{i=1}^{m} p_i(1-p_i)\right] \tag{25}$$

The optimal number of samples for each failure mode can be computed adaptively using the following procedure:

1) Estimate $p_i$ using a small number of Monte Carlo samples or other efficient strategy.
2) Estimate the total number of samples using Eq. (25).
3) Compute the optimal number of samples $n_i$ for each member failure mode using Eq. (17).
4) Compute $p_i$ and $P_f$ using Monte Carlo sampling with estimated optimal $n_i$.
5) Repeat steps 2–4 until $\Delta N = \Delta \sum n_i \le \varepsilon$.

## Application of Adaptive Optimal Sampling to Zone-Based System

The adaptive optimal sampling method is illustrated for the aircraft rotor disk shown in Fig. 5. The disk is discretized into 64 zones, that is, it is modeled as a series system consisting of 64 members. A fracture mechanics-based limit state is applied to each zone, and three primary random variables are used to model the uncertainties associated with material properties, applied stress, and initial anomaly size. Individual $p_i$ values are relatively small, and so Eq. (25) can be used to estimate the optimal number of samples for the system with a 10% sampling error and 95% confidence interval, that is, $k_{\alpha/2} = 1.96$ and $\gamma = 0.10$. Complete details regarding the disk, including values of the associated deterministic and random variables and number of samples applied to the zones, are provided in Ref. 18.

The following methods are used to estimate initial $p_i$ values before sampling: 1) uniform sampling, with the same number of samples in each zone, that is, 100, 1000, and 10,000 samples per zone; 2) first failure, where sampling is performed in a zone until either the limit state is violated or a specified number of samples (100 or 1000 samples per zone) have been applied and where $p_i$ is estimated as one divided by the number of samples to first failure, and 3) critical defect, where $p_i$ is estimated directly assuming that the dominant variable (initial anomaly size) is random and the remaining variables are deterministic. (See Ref. 4 for details.) For methods 1 and 2, $p_i$ is set equal to $1/n_i$ if no limit state violations occur within $n_i$ for a given zone.

The influence of the initial $p_i$ estimate on the predicted probability of failure is shown in Figs. 6–8 for a relatively weak zone, a strong zone, and the disk, respectively. In Fig. 6, the error in the initial $p_i$ estimate for a weak zone is between 1 and 29% depending on the method used, but is within 8% for the converged solution,

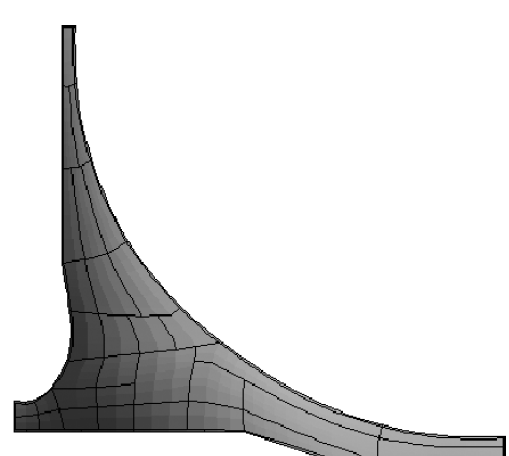

**Fig. 5 Adaptive optimal sampling methodology for aircraft rotor disk; see Ref. 18 for details.**

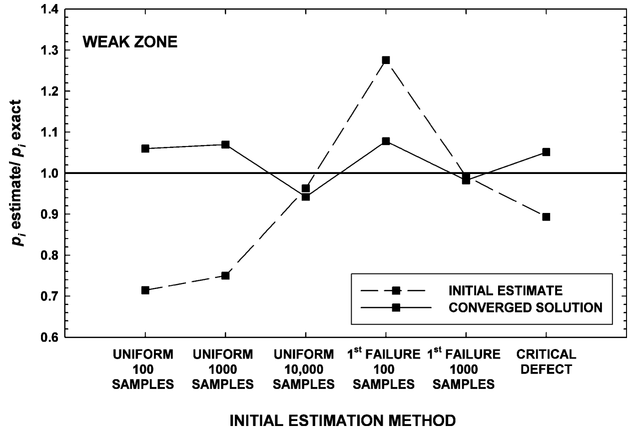


**Fig. 6 Influence of initial $p_i$ estimates on failure probability predictions for relatively weak zone.**

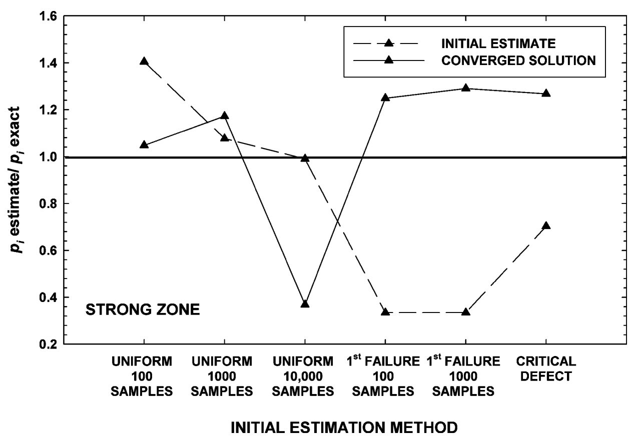


**Fig. 7 Influence of initial $p_i$ estimates on failure probability predictions for relatively strong zone.**

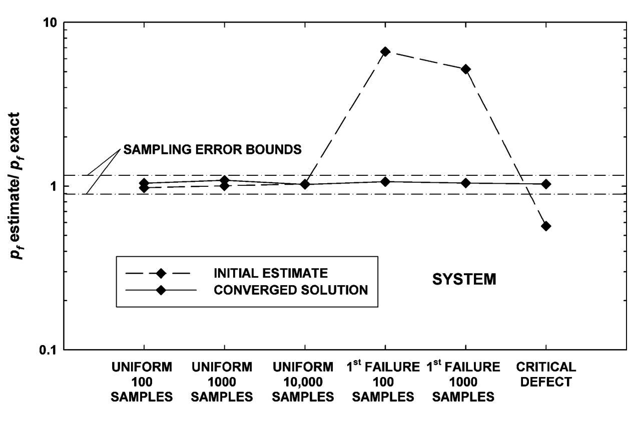


**Fig. 8 Influence of initial $p_i$ estimates on failure probability prediction of disk $P_f$ (system).**

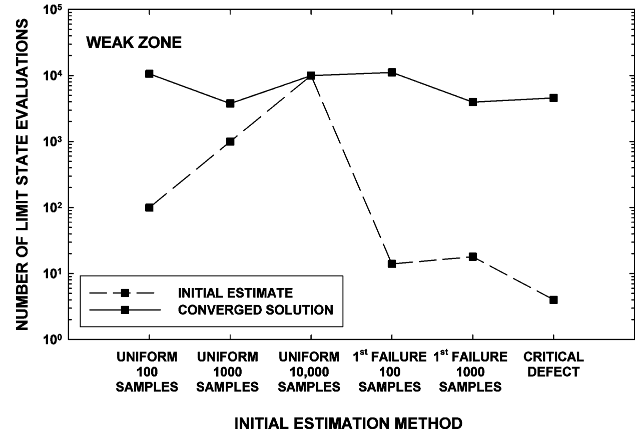


**Fig. 9 Influence of initial $p_i$ estimates on number of limit state evaluations required to accurately predict failure probability for relatively weak zone.**

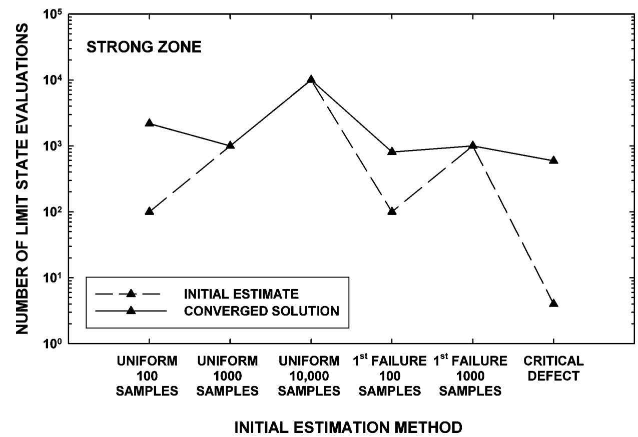


**Fig. 10 Influence of initial $p_i$ estimates on number of limit state evaluations required to accurately predict failure probability for relatively strong zone.**

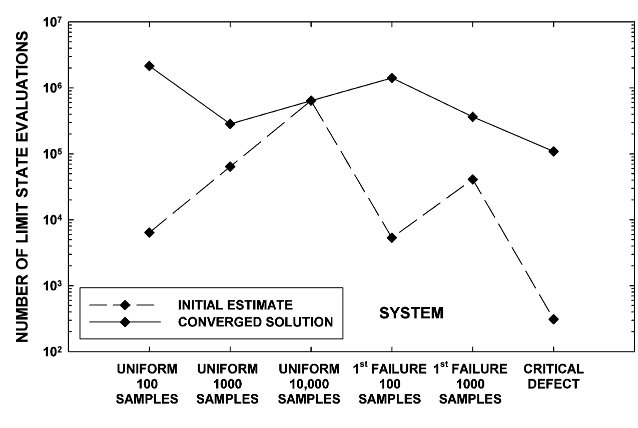


**Fig. 11 Influence of initial $p_i$ estimates on number of limit state evaluations required to accurately predict disk (system) failure probability.**

that is, after the adaptive optimal sampling algorithm is applied. In Fig. 7 it is shown that the sampling error for a strong zone is 1–67% for the initial estimate and 5–65% for the converged solution. Because the strong zone has a small influence on $P_f$, the relatively large postconvergence sampling error associated with some of the methods has a relatively small impact on the error associated with $P_f$. The sampling error for the system is shown in Fig. 8, with values of 0.5–660% and 2–8% for the initial and converged estimates, respectively. As shown in Fig. 8, all converged $P_f$ values fall within the 10% error bounds at 95% confidence, regardless of the method used for the initial prediction of $p_i$ values. This result suggests that sampling accuracy is somewhat independent of the method used to estimate initial $p_i$ values, provided that converged values are used to compute $P_f$.

The influence of the method used to estimate $p_i$ on computational efficiency is shown in Figs. 9–11 for a weak zone, a strong zone, and the system, respectively. For the weak zone (Fig. 9), a wide range of values for the number of samples, that is, limit state evaluations, is required to estimate $p_i$ for the various methods considered. However, converged solutions require a similar number of samples ranging from 3774 (uniform sampling with 1000 samples) to 10,600 (uniform sampling with 100 samples per zone). For the strong zone (Fig. 10), the range of samples for converged solutions ranges from 591 (critical defect) to 10,000 (uniform sampling with 10,000 samples per zone). The initial estimate appears to have a significant influence on the sampling efficiency of strong zones, with the most efficient methods requiring the fewest samples for the initial estimate. The number of samples associated with the system is shown in Fig. 11. For the converged solution, the number of samples ranges from about $1.0E5$ (critical defect) to over $2.0E6$ (uniform sampling with 100 samples per zone). These results indicate that sampling efficiency is highly dependent on the method used to estimate initial $p_i$ values.

## Summary

A methodology was presented for variance reduction of sampling-based series system reliability predictions based on optimal allocation of Monte Carlo samples to individual failure modes. An algorithm was presented for adaptively allocating samples to member failure modes based on initial estimates of the individual member failure probabilities. The optimal sampling methodology was demonstrated for a simple series system in which it was shown that the variance of the system failure probability is reduced compared to a uniform sampling approach due to the reduced variances in the failure probabilities associated with the weakest members. The adaptive optimal sampling methodology was illustrated for a gas-turbine engine disk modeled using several methods to estimate the failure probability $p_i$ in each zone before optimal sampling. For the example problem considered, it was shown that the computational accuracy of the method does not appear to depend on the initial $p_i$ estimate, whereas the computational efficiency is highly dependent on the initial $p_i$ estimate. The results can be applied to improve the efficiency of sampling-based series system reliability predictions.

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## References

[1]"Aircraft Accident Report—United Airlines Flight 232 McDonnell Douglas DC-10-10 Sioux Gateway Airport, Sioux City, Iowa, July 19, 1989," National Transportation Safety Board, Rept. NTSB/AAR-90/06, Washington, DC, Nov. 1990.

[2]Aerospace Industries Association Rotor Integrity Subcommittee, "The Development of Anomaly Distributions for Aircraft Engine Titanium Disk Alloys," *Proceedings of the 38th Structures, Structural Dynamics, and Materials Conference*, AIAA, Reston, VA, 1997, pp. 2543–2553.

[3]"Advisory Circular—Damage Tolerance for High Energy Turbine Engine Rotors," Federal Aviation Administration, Rept. AC 33.14-1, U.S. Dept. of Transportation, Washington, DC, Jan. 2001.

[4]Leverant, G. R., "Turbine Rotor Material Design—Final Report," Federal Aviation Administration, Rept. DOT/FAA/AR-00/64, U.S. Dept. of Transportation, Washington, DC, May 2000.

[5]Leverant, G. R., McClung, R. C., Millwater, H. R., and Enright, M. P, "A New Tool for Design and Certification of Aircraft Turbine Rotors," *Journal of Engineering for Gas Turbines and Power*, Vol. 126, No. 1, 2003, pp. 155–159.

[6]McClung, R. C., Enright, M. P., Millwater, H. R., Leverant, G. R., and Hudak, S. J., "A Software Framework for Probabilistic Fatigue Life Assessment," *Probabilistic Aspects of Life Prediction*, ASTM STP 1450, American Society for Testing and Materials International, West Conshohocken, PA, 2004, pp. 199–215.

[7]Enright, M. P., Huyse, L., McClung, R. C., and Millwater, H. R, "Probabilistic methodology for Life Prediction of Aircraft Turbine Rotors," *Proceedings of the 9th Biennial ASCE Aerospace Division International Conference on Engineering, Construction and Operations in Challenging Environments (Earth & Space 2004)*, edited by R. B. Malla and A. Maji, American Society of Civil Engineers, Reston, VA, 2004, pp. 453–460.

[8]McClung, R. C., Enright, M. P., Lee, Y.-D., Huyse, L., and Fitch, S. H. K., "Efficient Fracture Design for Complex Turbine Engine Components," *Proceedings of the 49th ASME International Gas Turbine & Aeroengine Technical Congress*, Paper GT2004-53323, American Society of Mechanical Engineers, June 2004.

[9]Hendawi, S., and Frangopol, D. M., "System Reliability and Redundancy in Structural Design and Evaluation," *Structural Safety*, Vol. 16, No. 1, 1994, pp. 47–71.

[10]Enright, M. P., and Frangopol, D. M., "Failure Time Prediction of Deteriorating Fail-Safe Structures," *Journal of Structural Engineering*, Vol. 124, No. 12, 1998, pp. 1448–1457.

[11]Ditlevsen, O., "Narrow Reliability Bounds for Structural Systems," *Journal of Structural Mechanics*, Vol. 7, No. 4, 1979, pp. 453–472.

[12]Melchers, R. E., *Structural Reliability Analysis and Prediction*, Wiley, New York, 1987, pp. 132–182.

[13]Mahadevan, S., and Raghothamachar, P., "Adaptive Simulation for System Reliability Analysis of Large Structures," *Computers and Structures*, Vol. 77, No. 6, 2000, pp. 725–734.

[14]Au, S. K., and Beck, J. L., "First Excursion Probabilities for Linear Systems by Very Efficient Importance Sampling," *Probabilistic Engineering Mechanics*, Vol. 16, No. 3, 2001, pp. 193–207.

[15]Mori, Y., and Kato, T., "Multinormal Integrals by Importance Sampling for Series System Reliability," *Structural Safety*, Vol. 25, No. 4, 2003, pp. 363–378.

[16]Wu, Y. T., Enright, M. P., and Millwater, H. R.,"Probabilistic Methods for Design Assessment of Reliability with Inspection," *AIAA Journal*, Vol. 40, No. 5, 2002, pp. 937–946.

[17]Millwater, H. R., Enright, M. P., and Fitch, S. K., "A Convergent Probabilistic Technique for Risk Assessment of Gas Turbine Disks Subject to Metallurgical Defects," AIAA Paper 2002-1382, April 2002.

[18]Huyse, L., and Enright, M. P., "Efficient Statistical Analysis of Failure Risk in Engine Rotor Disks Using Importance Sampling Techniques," AIAA Paper 2003-1838, April 2003.

[19]Millwater, H. R., Shook, B. D., Guduru, S., and Constantinides, G., "Application of Parallel Processing Methods to Probabilistic Fracture Mechanics Analysis of Gas Turbine Disks," AIAA Paper 2004-1745, April 2004.

[20]Wu, Y.-T., Enright, M. P., and Millwater, H. R., "Efficient and Accurate Methods for Probabilistic Analysis of Titanium Rotors," *Proceedings of the 8th ASCE Joint Specialty Conference on Probabilistic Mechanics and Structural Reliability*, edited by A. Kareem, A. Haldar, B. Spencer, and E. Johnson, Paper PMC2000-221, American Society of Civil Engineers, Reston, VA, 2000.

[21]Enright, M. P., and Millwater, H. R., "Optimal Sampling Techniques for Zone-Based Probabilistic Fatigue Life Prediction," AIAA Paper 2002-1383, April 2002.

[22]Ramirez-Marquez, J. E., Coit, D. W., and Jin, T., "Test Plan Allocation to Minimize System Reliability Estimation Variability," *International Journal of Reliability, Quality, and Safety Engineering*, Vol. 11, No. 3, 2004, pp. 257–272.

[23]Ang, A. H.-S., and Tang, W. H., *Probability Concepts in Engineering Planning and Design*, Wiley, New York, 1975, pp. 252–254.

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